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Asymptotic behaviour of a scalar in an axisymmetric final period turbulent wake

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The behaviour of a conserved scalar field in the final period of an axisymmetric turbulent wake is investigated theoretically. Explicit formulae are derived for the behaviour, during and just prior to the scalar field final period, of quantities such as mean concentration, concentration correlations and velocity–scalar or vorticity–scalar cross-correlations. In particular, if τ is the time measured from a virtual origin it is shown that scalar intensity, which decays as $\tau^{-\frac{5}{2}}$, is more persistent asymptotically than turbulence intensity, which displays a $\tau^{-\frac{7}{2}}$ decay. By deriving the scalar field structure as perturbed by the Reynolds stresses acting on mean scalar gradients in the final period it is shown, e.g., that the mean field perturbation dies out at least as rapidly as τ^{-4}, whereas the mean field itself decays as τ^{-1}. Numerical results are presented to display the spatial structure of typical scalar correlations and velocity–scalar cross-correlations, which are compared where possible with non-asymptotic measurements in the wake of a heated sphere.

1. Introduction

In this paper we explore the possibility of deducing the structure of a passive scalar field in the final period regime of an axisymmetric turbulent wake. The wake is considered to be embedded in an infinite fluid. This particular non-homogeneous turbulent field falls in a class for which the final period of decay was analysed by Phillips (1955), who deduced the velocity field in terms of two invariants of the wake motion: the net linear momentum and the net angular momentum imparted to the fluid by a localized disturbance. In particular, he derived expressions for the energy spectrum tensor in the final period of an axisymmetric wake.

Non-homogeneity is the normal condition for most manifestations of turbulence, and those scalar processes that depend significantly on large-scale turbulent motions are influenced by both the production of scalar intensity due to interaction with the mean fields, and by the transfer of turbulent energy and scalar intensity from one locality of the turbulence to another.

Boundaries may also play a significant role in modifying decay processes. However, in the situation considered here the boundaries are assumed to be so far removed from regions where there is appreciable turbulence or scalar intensity that their influence can be neglected. Furthermore, in the final period of a wake, the mean flow exercises an ever-decreasing influence on the turbulence structure.

We shall show that, asymptotically, the dominant processes are diffusion to regions of lower intensity and interactions between the mean scalar field and the turbulence, and between the mean velocity field and the scalar fluctuations. The physical significance of the last of these three processes is of limited interest, involving in the final period only the distortion of the scalar structure due to advection by a weak, variable mean velocity.

The investigation which follows concentrates on those modifications to the scalar field that are consequences of the other two processes described above, and occur ubiquitously in the decay of scalar fields in non-homogeneous turbulence. In the relatively simple situation studied here, we have the advantage of an analytical statement of the velocity field structure (Phillips 1955). No turbulence measurements in an axisymmetric, heated wake have been taken below $R_\lambda = 16$ (Freymuth & Uberoi 1973); but these authors are presently attempting to extend their technique closer to the asymptotic, diffusion-dominated regime.

2. Scalar field moment equations

There exists a fairly extensive literature concerned with the measurement of scalar quantities (Freymuth & Uberoi 1971, 1973; Gibson, Chen & Lin 1968) and especially temperature in a turbulent wake, and the general one- and two-point moment equations for an advected scalar in a turbulent shear field are known (Corrsin 1952).

Let $\Gamma(\mathbf{x}, t)$ represent the random scalar field at position $\mathbf{x}$ at time t. $\Gamma(\mathbf{x}, t)$ is introduced into the wake during an initial period of development, and it is then free to be advected and diffused without additional scalar matter being added or removed. Let $\mathbf{U}(\mathbf{x}, t)$ represent an incompressible turbulent wake produced asymptotically when a solid of revolution moves at high Reynolds number in the direction of its axis through a large body of fluid originally at rest. Following Phillips, we adopt a system of co-ordinates such that the 1 axis lies along the wake axis. Furthermore, we assume with him that the turbulence and the scalar field are in a uniform state of decay along lengths of the wake over which correlations are significant. Formally, this assumption of a modified, axial, statistical homogeneity removes from the equations axial derivatives of single-point moments, and reduces the mean quantities to being undirectional and functions of only transverse co-ordinates. Thus, if mean quantities are denoted by bars and fluctuations by lower case letters, we have

$$U_i(\mathbf{x}, t) = \overline{U}_i(\mathbf{x}, t) + u_i(\mathbf{x}, t)$$

and

$$\Gamma(\mathbf{x}, t) = \overline{\Gamma}(\mathbf{x}, t) + \gamma(\mathbf{x}, t).$$

With the assumptions noted above, the mean fields are reduced to

$$\overline{U}_i(\mathbf{x}, t) = \overline{U}(x_2, x_3, t)\, \delta_{i1}, \tag{1}$$

$$\overline{\Gamma}(\mathbf{x}, t) = \overline{\Gamma}(x_2, x_3, t), \tag{2}$$

where $\mathbf{x} = (x_1, x_2, x_3)$. The behaviour of $\Gamma(\mathbf{x}, t)$ is described by

$$\frac{\partial \Gamma}{\partial t} + \frac{\partial}{\partial x_i}(u_i \Gamma) = D\nabla^2 \Gamma, \tag{3}$$

where D is the scalar diffusivity assumed independent of position and time.

On forming moment equations in the usual manner, we obtain from (3), using (1) and (2),

$$\left\{\frac{\partial}{\partial t} - D\nabla^2\right\} \overline{\Gamma} = -\frac{\partial}{\partial x_i} \overline{i_i \gamma}, \tag{4}$$

$$\left\{\frac{\partial}{\partial t} - D(\nabla^2 + \nabla'^2) + (\overline{U}' - \overline{U})\frac{\partial}{\partial r_1}\right\} \overline{\gamma\gamma'}$$
$$= -\overline{u_i \gamma'}\frac{\partial \overline{\Gamma}}{\partial x_i} - \overline{u_i' \gamma}\frac{\partial \overline{\Gamma}'}{\partial x_i'} - \frac{\partial}{\partial x_k'}\overline{u_k' \gamma' \gamma} - \frac{\partial}{\partial x_k}\overline{u_k \gamma \gamma'}, \tag{5}$$

$$\left\{\frac{\partial}{\partial t} - (D\nabla^2 + \nu\nabla'^2) + (\overline{U}' - \overline{U})\frac{\partial}{\partial r_1}\right\} \overline{u_i \gamma'} = \overline{u_k' u_i}\frac{\partial \overline{\Gamma'}}{\partial x_k'} - \frac{\partial}{\partial x_k'}(\overline{u_k' u_i \gamma'}), \tag{6}$$

where the asymptotic form of the Navier–Stokes equations has been employed: $\partial u_i/\partial t = \nu\, \partial^2 u_i/\partial x_j\, \partial x_j$, ν is the kinematic viscosity of the fluid taken as constant and the prime represents a spatial location $\mathbf{x}'$ rather than $\mathbf{x}$. For example, the operator ∇'^2 signifies the Laplacian

$$\nabla'^2 = \frac{\partial^2}{\partial x_j'\, \partial x_j'}.$$

Furthermore, $r_1 = x_1' - x_1$ and a modified statistical homogeneity in the axial direction has been used. Thus

$$\overline{\gamma\gamma'} = \overline{\gamma\gamma'}(r_1, x_2, x_3, x_2', x_3', t),$$
$$\overline{u_i\gamma'} = \overline{u_i\gamma'}(r_1, x_2, x_3, x_2', x_3', t),$$
$$\overline{\Gamma} = \overline{\Gamma}(x_2, x_3, t).$$

The set of equations (4)–(6) is not closed, and the specification of initial data for $\overline{\Gamma}$, $\overline{\gamma\gamma'}$ and $\overline{u_i\gamma'}$ is not sufficient information, in general, to solve for the time evolution of these quantities. In § 3 we consider the asymptotic decay regime where elementary solutions can be readily obtained.

3. The final period of decay of the scalar field

Consider an infinite body of fluid which initially does not contain any amount of the scalar field Γ. Let Γ be introduced locally at some region of the fluid, so that the total intensity of the scalar is finite. Further, we assume the fluid was originally at rest and subjected to a local disturbance with non-zero linear momentum. Under these circumstances, since neighbouring fluid elements can only acquire Γ by molecular diffusion, the concentration field at large distances from the disturbed regions will decay exponentially. The Fourier transform for $\Gamma(\mathbf{x}, t)$ is defined by

$$\Gamma(\mathbf{x}, t) = \int C(\mathbf{k}, t) \exp\{i\mathbf{k}, \mathbf{x}\}\, d\mathbf{k}, \tag{7}$$

where $d\mathbf{k}$ is the volume element $dk_1\,dk_2\,dk_3$ about wavenumber vector $\mathbf{k}$. The asymptotic decay behaviour of Γ in $\mathbf{x}$ space guarantees the continuity of $C(\mathbf{k},t)$ and all its derivatives with respect to $\mathbf{k}$. An expansion of $C(\mathbf{k},t)$ about the origin in wavenumber space leads to

$$C(\mathbf{k},t) = C + k_j C^j + k_i k_j C^{ij} + O(k^3), \tag{8}$$

where the coefficients C, C^j, etc. are single-valued functions of time and the particular realization of the flow field.

If the scalar field Γ is neither created nor destroyed after it has been introduced into the fluid, it follows from the inverse of (7) evaluated at $\mathbf{k} = 0$ that the total scalar content

$$\int \Gamma(\mathbf{x},t)\,d\mathbf{x}$$

is conserved and equals C, which is now specifically time-independent, and represents an invariance for the scalar field. In the final period of decay the behaviour of the scalar field is governed by (3), with the convective term neglected:

$$\frac{\partial}{\partial t}\Gamma(\mathbf{x},t) = D\nabla^2\Gamma(\mathbf{x},t).$$

The Fourier transform $C(\mathbf{k},t)$ clearly obeys the corresponding equation

$$\frac{\partial}{\partial t}C(\mathbf{k},t) = Dk^2 C(\mathbf{k},t),$$

which exhibits the solution

$$C(\mathbf{k},t) = C(\mathbf{k},t_0)\exp-\{Dk^2(t-t_0)\}, \tag{9}$$

where t_0 is a virtual time origin. For asymptotically large values of $(t-t_0)$, the dominant term in (9) is given by the first term in the expansion (8):

$$C(\mathbf{k},t) \doteq C\exp-\{Dk^2(t-t_0)\}.$$

On applying this result to (7), we find

$$\Gamma(\mathbf{x},t) = C\left\{\frac{\pi}{D(t-t_0)}\right\}^{\frac{3}{2}}\exp-\left\{\frac{\mathbf{x}.\mathbf{x}}{4D(t-t_0)}\right\}. \tag{10}$$

The linearity of (1) guarantees that any superposition of solutions such as (10) above is also a solution.

For the construction of the final period scalar decay in turbulence, it is appropriate to consider the scalar field formed by a continuous distribution of C such that at the point $\boldsymbol{\chi}$ the density is $C(\boldsymbol{\chi})$. An element of $\boldsymbol{\chi}$ space $\delta\boldsymbol{\chi}$ thus produces a concentration $\delta\Gamma(\mathbf{x},t)$ given by

$$\delta\Gamma(\mathbf{x},t) = C(\boldsymbol{\chi})\,\delta\boldsymbol{\chi}\left\{\frac{\pi}{D(t-t_0)}\right\}^{\frac{3}{2}}\exp-\left\{\frac{(\mathbf{x}-\boldsymbol{\chi})^2}{4D(t-t_0)}\right\}.$$

The entire concentration field is specified by the distribution of $C(\boldsymbol{\chi})$,

$$\Gamma(\mathbf{x},t) = \left\{\frac{\pi}{D(t-t_0)}\right\}^{\frac{3}{2}}\int C(\boldsymbol{\chi})\exp-\left\{\frac{(\mathbf{x}-\boldsymbol{\chi})^2}{4D(t-t_0)}\right\}d\boldsymbol{\chi}. \tag{11}$$

The general solution to (1) can be expressed in the form

$$\Gamma(\mathbf{x},t) = \frac{1}{[4\pi D(t-t_0)]^{\frac{3}{2}}}\int \Gamma(\mathbf{x}',t_0)\exp-\left\{\frac{(\mathbf{x}-\mathbf{x}')^2}{4D(t-t_0)}\right\}d\mathbf{x}'.$$

Thus the behaviour of *any* scalar field, at all instances in the final period, can be specified by an appropriate density distribution $C(\chi)$, which can be identified as, for example,

$$C(\chi) = \Gamma(\mathbf{x}',t_0)/2\pi^3.$$

Using a device previously applied to the velocity field, we can define a line distribution of scalar content $C(\chi_1)$, which will approximate the volume distribution in the case of the long narrow wake under investigation here. Define

$$C(\chi_1) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C(\chi)\,d\chi_2\,d\chi_3,$$

and (11) becomes

$$\Gamma(\mathbf{x},t) = \left\{\frac{\pi}{D(t-t_0)}\right\}^{\frac{3}{2}}\exp-\left\{\frac{x_2^2+x_3^2}{4D(t-t_0)}\right\}\int_{-\infty}^{+\infty} C(\chi_1)\exp-\left\{\frac{(x_1-\chi_1)^2}{4D(t-t_0)}\right\}d\chi_1. \quad (12)$$

The mean field $\overline{\Gamma}(\mathbf{x},t)$ can be obtained from (12) by noting that $\overline{C}(\chi_1)$ must be a constant by axial homogeneity. Therefore,

$$\overline{\Gamma}(\mathbf{x},t) = \overline{C}\left\{\frac{\pi}{D(t-t_0)}\right\}^{\frac{3}{2}}\exp-\left\{\frac{x_2+x_3^2}{4D(t-t_0)}\right\}\int_{-\infty}^{+\infty}\exp-\left\{\frac{y^2}{4D(t-t_0)}\right\}dy,$$

from which we deduce that

$$\overline{\Gamma}(\mathbf{x},t) = 2\overline{C}\frac{\pi^2}{D(t-t_0)}\exp-\left\{\frac{x_2^2+x_3^2}{4D(t-t_0)}\right\}. \quad (13)$$

For later convenience, we can write down the Fourier transform of $\overline{\Gamma}(\mathbf{x},t)$ defined by, say, $J(\mathbf{k},t)$. Then

$$J(\mathbf{k},t) = 2\overline{C}\pi\exp-\{D(t-t_0)(k_2^2+k_3^2)\}. \quad (14)$$

From (12) it is also possible to derive the form of moments of the concentration field in the final period.

The concentration correlation

As an example we may consider the product $\Gamma(\mathbf{x},t)\,\Gamma(\mathbf{x}',t)$ which, from (12), can be written as

$$\Gamma(\mathbf{x},t)\,\Gamma(\mathbf{x}',t) = \left\{\frac{\pi}{D(t-t_0)}\right\}^3\exp-\left\{\frac{x_i^2+x_i'^2}{4D(t-t_0)}\right\}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C(\chi_1)\,C(\chi_1')$$

$$\times\exp-\left\{\frac{(x_1-\chi_1)^2+(x_1'-\chi_1')^2}{4D(t-t_0)}\right\}d\chi_1\,d\chi_1' \quad (i = 2,3).$$

On taking an ensemble average and defining the correlation of the fluctuations of C by

$$h(\chi_1'-\chi_1) = \overline{C(\chi_1')\,C(\chi_1)} - \overline{C(\chi_1)}^2,$$

we have

$$\overline{\gamma(\mathbf{x},t)\,\gamma(\mathbf{x}',t)} = \left\{\frac{\pi}{D(t-t_0)}\right\}^3 \exp -\tfrac{1}{4}(\xi_i^2+\xi_i'^2)\,Pr\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\sigma) \times \exp -\left\{\frac{[y^2+(x_1'-x_1-y-\sigma)^2]}{4D(t-t_0)}\right\} dy\,d\sigma,$$

where $\xi_i = x_i[D(t-t_0)]^{-\frac{1}{2}}$ $(i = 2,3)$, $\sigma = \chi_1'-\chi_1$, $y = \chi_1 - x_1$ and $Pr = \nu/D$. Further, let

$$\eta = \frac{x_1'-x_1-\sigma}{[D(t-t_0)]^{\frac{1}{2}}}.$$

Then

$$\overline{\gamma(\mathbf{x},t)\,\gamma(\mathbf{x}',t)} = \frac{2^{\frac{1}{2}}\pi^{\frac{7}{2}}}{[D(t-t_0)]^{\frac{5}{2}}}\int_{-\infty}^{+\infty} h(\sigma)\exp\{-\eta^2/8\}\,d\sigma \exp\{-\tfrac{1}{4}[\xi_i^2+\xi_i'^2]\}\,Pr.$$

An expansion of $\exp -\{\frac{1}{8}n^2\}$ in a series whose coefficients involve increasing inverse powers of $(t-t_0)$ reduces the asymptotic form of the correlation to

$$\overline{\gamma(\mathbf{x},t)\,\gamma(\mathbf{x}',t)} = \frac{2^{\frac{1}{2}}\pi^{\frac{7}{2}}H}{[D(t-t_0)]^{\frac{5}{2}}}\exp -\{[\boldsymbol{\xi}^2+\boldsymbol{\xi}'^2+\tfrac{1}{2}\xi_1^2]\,Pr\},$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are the two-dimensional vectors (ξ_2,ξ_3) and (ξ_2',ξ_3'),

$$\xi_1 = \frac{x_1'-x_1}{[\nu(t-t_0)]^{\frac{1}{2}}} \quad \text{and} \quad H = \int_{-\infty}^{+\infty} h(\sigma)\,d\sigma.$$

To be concise we will normally write $\overline{\gamma(x,t)\,\gamma(x',t)}$ as $\overline{\gamma\gamma'}$. The Fourier transform of this correlation, defined by

$$\phi(k_1,\mathbf{k},\mathbf{k}',t) = (2\pi)^{-5}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{\gamma\gamma'}(r_1,\mathbf{x},\mathbf{x}',t) \times \exp - i\{k_1 r_1+\mathbf{k}.\mathbf{x}+\mathbf{k}'.\mathbf{x}'\}\,dr_1\,d\mathbf{x}\,d\mathbf{x}'$$

can easily be shown to be

$$\phi(k_1,\mathbf{k},\mathbf{k}',t) = 2^3\pi^{\frac{7}{2}}H\exp -\{(k^2+k'^2+2k_1)\,D(t-t_0)\}, \tag{15}$$

where $\mathbf{k}$, $\mathbf{k}'$ here, and subsequently, are the two-dimensional vectors (k_2,k_3) and (k_2',k_3'), respectively.

It is also possible to make use of the asymptotic solution (10) and analogous results for the velocity field $u_i(\mathbf{x},t)$ to obtain expressions for the energy tensor and for such moments as $\overline{u_i\gamma'}$, $\overline{u_i\gamma\gamma'}$ and $\overline{w_i\gamma'}$, where $\mathbf{w}$ is the fluctuating vorticity at a point. Phillips showed that a localized disturbance with non-zero total momentum, acting on an infinite fluid otherwise at rest, produces the asymptotic velocity and vorticity fields

$$u_i(\mathbf{x},t) = 4\left\{\frac{\pi}{\nu(t-t_0)}\right\}^{\frac{3}{2}}\frac{M_j}{\xi^3}\left\{\left(\frac{3\xi_i\xi_j}{\xi^2}-\delta_{ij}\right)\operatorname{erf}\tfrac{1}{2}\xi + \frac{1}{4}\left[(2+\xi^2)\,\delta_{ij}-(\xi^2+6)\frac{\xi_i\xi_j}{\xi^2}\right]\xi\exp\{-\tfrac{1}{4}\xi^2\}\right\},$$

$$w_i(\mathbf{x},t) = \frac{\pi^{\frac{3}{2}}}{2[\nu(t-t_0)]^2}\,\epsilon_{ijk}M_j\xi_k\exp\{-\tfrac{1}{4}\xi^2\},$$

where $\text{erf}\,\frac{1}{2}\xi = \int_0^{\frac{1}{2}\xi} \exp\{-y^2\}\,dy$ and M_j is a constant, proportional to the total momentum of the disturbance, i.e. (Phillips 1955) $M_j = (2\pi)^{-3}\int u_j(\mathbf{x})\,d\mathbf{x}$.

The energy spectrum tensor

The motion in the asymptotic wake was shown (Phillips 1955) to consist of a superposition of a mean streaming velocity $\overline{U}(x_2, x_3, t)$ and a fluctuating velocity field, for which were displayed the diagonal terms of the energy spectrum tensor $\phi_{ij}(k_1, \mathbf{k}, \mathbf{k}', t)$. These expressions, which are reproduced below (with a minor correction in sign) along with the non-diagonal elements of the energy spectrum tensor, have, as scalar coefficients, integrals of the linear momentum correlations along the wake axis. Phillips also showed that, for a long narrow wake, the motion, which depends in principle on the volume distribution of M_j, can be represented adequately by the line distribution $m_j(\chi_1)$, where the 1 axis lies along the axis of the wake. The appropriate relationship is of course

$$m_j(\chi_1) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M_j(\boldsymbol{\chi})\,d\chi_2\,d\chi_3.$$

Further define (with the aid of axial homogeneity and axisymmetry)

$$f(\sigma) = \overline{m_1(\chi_1)\,m_1(\chi_1+\sigma)},$$

and $$g(\sigma) = \overline{m_2(\chi_1)\,m_2(\chi_1+\sigma)} = \overline{m_3(\chi_1)\,m_3(\chi_1+\sigma)}.$$

These functions were shown to have the property

$$\int_{-\infty}^{+\infty} f(\sigma)\,d\sigma = \int_{-\infty}^{+\infty} g(\sigma)\,d\sigma = 0,$$

and to appear in the energy spectrum tensor in the form of the constants

$$F = -\int_{-\infty}^{+\infty} \sigma^2 f(\sigma)\,d\sigma, \quad G = -\int_{-\infty}^{+\infty} \sigma^2 g(\sigma)\,d\sigma.$$

The energy spectrum tensor elements then become

$$\begin{aligned}
\phi_{11}(K_1, \mathbf{K}, \mathbf{K}', t) &= D_1[K_1^2K^2K'^2F - K_1^4(K_2K_2' + K_3K_3')\,G],\\
\phi_{22}(K_1, \mathbf{K}, \mathbf{K}', t) &= D_1[\{K_1^6 - K_1^4(K_3^2 + K_3'^2) + K_1^2K_3K_3'(K_2K_2' + KK_3')\}\,G\\
&\quad - K_1^4K_3K_3'F],\\
\phi_{33}(K_1, \mathbf{K}, \mathbf{K}', t) &= D_1[\{K_1^6 - K_1^4(K_2^2 + K_2'^2) + K_1^2K_2K_2'(K_3K_3' + K_2K_2')\}\,G\\
&\quad - K_1^4K_2K_2'F],\\
\phi_{12}(K_1, \mathbf{K}, \mathbf{K}', t) &= D_1[K_1^3K_2'K^2F - \{K_1^5K_2 + K_1^3K_3'(K_3K_2' - K_2K_3')\}\,G],\\
\phi_{13}(K_1, \mathbf{K}, \mathbf{K}', t) &= D_1[K_1^3K_2'K^2F - \{K_1^5K_2 + K_1^3K_2'(K_2K_3' - K_3K_2')\}\,G],\\
\phi_{23}(K_1, \mathbf{K}, \mathbf{K}', t) &= D_1[\{K_1^4(K_2'K_3' + K_3K_2) - K_1^2K_3K_2'(K_2K_2' + K_3K_3')\}\,G\\
&\quad - K_1^4K_3'K_2F],
\end{aligned} \tag{16}$$

where K_1, $\mathbf{K}$ and $\mathbf{K}'$ are the normalized Fourier transform variables corresponding to ξ_1, $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$, and

$$D_1 = \frac{\pi}{[\nu(t-t_0)]^{\frac{7}{2}}} \frac{\exp-\{2K_1^2+K^2+K'^2\}}{(K_1^2+K^2)(K_1^2+K'^2)}.$$

Similarly, the asymptotic mean flow is shown to have the form

$$\overline{U}(\xi_2, \xi_3, t) = \frac{2\pi^2\overline{m}_1}{\nu(t-t_0)} \exp\{-\tfrac{1}{4}\xi_i\xi_i\}, \tag{17}$$

where $\overline{m}_1$ is proportional to the mean linear momentum, which is entirely in the 1 direction.

The velocity–concentration cross-correlation

Mixed velocity field–scalar field correlations in the final period can be obtained by combining (10) and (16) or (17) and using a superposition technique analogous to that which led to (15). It turns out to be convenient first to derive the vorticity–scalar cross moment $\overline{w_i\gamma'}$ and to construct $\overline{u_i\gamma'}$ from it. Symmetry considerations show

$$\overline{m_j(\chi_1)\, C(\chi_1')} = \delta_{1j}\, e(\sigma),$$

where as before $\sigma = \chi_1' - \chi_1$ and $e(\sigma)$ satisfies

$$\int_{-\infty}^{+\infty} e(\sigma)\, d\sigma = 0.$$

Then $\overline{w_i\gamma'}$ can be shown to be

$$\overline{w_i\gamma'}(\xi_1, \xi, \xi', t) = \frac{\alpha^{\frac{3}{2}} Pr\, \xi_1 \pi^{\frac{3}{2}} E}{4[\nu(t-t_0)]^{\frac{7}{2}}} \exp\{-\tfrac{1}{4}(\xi^2 + Pr\, \xi'^2 + \alpha\xi_1^2)\}\,(0, -\xi_3, \xi_2), \tag{18}$$

where
$$E = \int_{-\infty}^{+\infty} \sigma e(\sigma)\, d\sigma, \quad \alpha = \frac{Pr}{1+Pr}.$$

The interpretation of (18) is particularly simple in cylindrical co-ordinates where only the azimuthal component $\overline{w_\theta\gamma'}$ is non-zero:

$$\overline{w_\theta\gamma'}(\xi_1, \boldsymbol{\xi}, \boldsymbol{\xi}', t) = \frac{\alpha^{\frac{3}{2}} Pr\, \xi_1 \pi^{\frac{3}{2}} E}{4[\nu(t-t_0)]^{\frac{7}{2}}}\, \xi \exp\{-\tfrac{1}{4}(\xi^2 + Pr\, \xi'^2 + \alpha\xi_1^2)\}. \tag{19}$$

Since the velocity field is at rest at infinity the vector $\overline{u_i\gamma'}$ is entirely solenoidal, and it can be computed from $\overline{w_i\gamma'}$ as (Batchelor 1967)

$$\overline{u\gamma'}(\mathbf{x}, \mathbf{x}'') = -\frac{1}{4\pi}\int_{V(\mathbf{x}')} \frac{(\mathbf{x}-\mathbf{x}')\, x\overline{\mathbf{w}\gamma'}(\mathbf{x}', \mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^3}\, dV(x\mathbf{x}'), \tag{20}$$

where $\mathbf{x}$, $\mathbf{x}'$ and $\mathbf{x}''$ are here three-dimensional space vectors. In component form we have

$$\overline{u_1\gamma'} = -\frac{1}{4\pi}\int_{V(\mathbf{x}')} \frac{[(x_2-x_2')\,\overline{w_3\gamma'}(\mathbf{x}', \mathbf{x}'') - (x_3-x_3')\,\overline{w_2\gamma'}(\mathbf{x}', \mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|^3}\, dV(\mathbf{x}'), \tag{21}$$

$$\overline{u_2\gamma'} = +\frac{1}{4\pi}\int \frac{(x_1-x_1')\,\overline{w_3\gamma'}(\mathbf{x}', \mathbf{x}'')}{|\mathbf{x}-\mathbf{x}'|^3}\, dV(\mathbf{x}'), \tag{22}$$

$$\overline{u_3\gamma'} = -\frac{1}{4\pi}\int \frac{[(x_1-x_1')\,\overline{w_2\gamma'}(\mathbf{x}', \mathbf{x}'')]}{|\mathbf{x}-\mathbf{x}'|^3}\, dV(\mathbf{x}'), \tag{23}$$

where $\overline{w_2\gamma'}$, $\overline{w_3\gamma'}$ are given by (18).

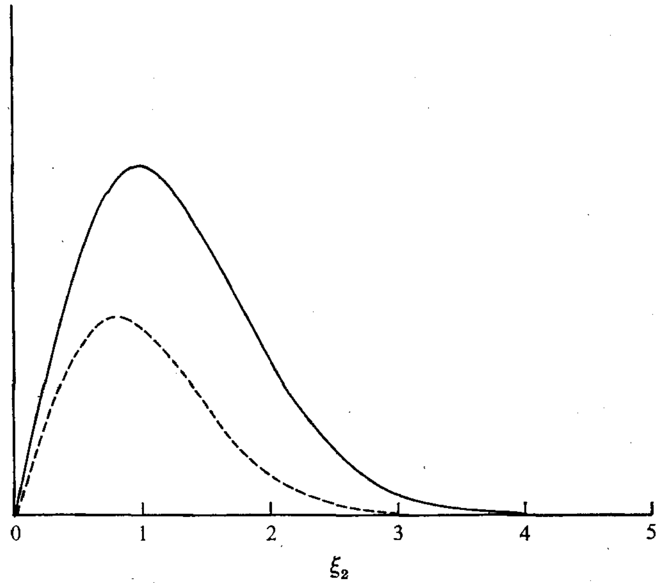

FIGURE 1. Velocity–scalar correlations in the final period: ——, $\overline{u_2\gamma}(0, \xi_2, 0, \xi_2, 0)$; - - - -, $\overline{u_2\gamma^2}(0, \xi_2, 0, \xi_2, 0)$.

Quantity	Asymptotic decay rate	Quantity	Asymptotic decay rate
$\overline{U}$	$(t-t_0)^{-1}$	$\overline{u_i\gamma'}$	$(t-t_0)^{-3}$
$\overline{\Omega}$	$(t-t_0)^{-\frac{3}{2}}$	$\overline{w_i\gamma'}$	$(t-t_0)^{-\frac{7}{2}}$
$\overline{\Gamma}$	$(t-t_0)^{-1}$	$\overline{\gamma\gamma'}$	$(t-t_0)^{-\frac{5}{2}}$
$\nabla\overline{U}$	$(t-t_0)^{-\frac{3}{2}}$	$\frac{\partial}{\partial x_j}\overline{u_iu'_j\gamma}$	$(t-t_0)^{-\frac{11}{2}}$
$\nabla\overline{\Omega}$	$(t-t_0)^{-2}$	$\frac{\partial}{\partial x_j}\overline{u_i\gamma\gamma'}$	$(t-t_0)^{-5}$
$\nabla\overline{\Gamma}$	$(t-t_0)^{-\frac{3}{2}}$		
$\overline{u_iu'_j}$	$(t-t_0)^{-\frac{7}{2}}$		
$\overline{w_iw'_j}$	$(t-t_0)^{-\frac{9}{2}}$		

TABLE 1. Final period time dependence

We have found it necessary to resort to numerical integration to obtain the vectors $\overline{u_i\gamma'}(\xi_1, \xi_2, \xi_3, \xi'_2, \xi'_3, t)$. In figure 1 the results for $\overline{u_2\gamma}(0, \xi_2, 0, \xi_2, 0, t)$ are presented, along with $\overline{u_2\gamma^2}(0, \xi_2, 0, \xi_2, 0, t)$. The ordinate scale, which depends on t, has been left arbitrary in both cases. The temporal dependence of various moments in the final period can be obtained quite simply by the methods developed in this section. Table 1 lists the asymptotic results for all correlations which occur in a moment formulation of the final period of turbulent wakes. In § 4, to obtain scalar field information in the asymptotic wake, we make use of these temporal decay results to develop a self-consistent perturbation procedure.

4. The penultimate period of decay of a scalar field

Under some circumstances, discussed below, it is possible to obtain scalar field decay results that are valid in a time regime prior to the final period of decay where, by definition, only molecular dissipative processes occur. In particular, the influence of the interaction of mean scalar gradient with both Reynolds stresses and turbulent scalar flux vectors can be incorporated. We have coined the term *penultimate period* to describe this temporal regime, which, when it exists, immediately precedes the final period.

From the asymptotic decay results tabulated in § 3 one can hope to deduce the structure of the scalar field at a time prior to that of its final period. The turbulence, which decays more rapidly than the scalar field and which may not necessarily have been generated at the same time, is still assumed to be in its final period of decay. For example, one can contemplate having chemical species generated by photochemical effects in the wake of a high-speed aircraft. In these circumstances (4)–(6) continue to be the appropriate description and, moreover, there is a formal simplification that can be employed to reduce the computational labour in solving them. If we use as independent variables the vector location of points moving with the local mean velocity, the defining equations exhibit the form

$$\frac{\partial\overline{\Gamma}}{\partial t} - D\nabla^2\overline{\Gamma} = -\frac{\partial}{\partial x_i}\overline{u_i\gamma}, \tag{24}$$

$$\left\{\frac{\partial}{\partial t} - D(\nabla^2+\nabla'^2)\right\}\overline{\gamma\gamma'} = -\overline{u_i\gamma'}\frac{\partial\overline{\Gamma}}{\partial x_i} - \overline{u_i'\gamma}\frac{\partial\overline{\Gamma}'}{\partial x_i'} - \frac{\partial}{\partial x_k'}\overline{u_k'\gamma'\gamma} - \frac{\partial}{\partial x_k'}u_k\gamma\gamma', \tag{25}$$

$$\left\{\frac{\partial}{\partial t} - D\nabla^2 - \nu\nabla'^2\right\}\overline{u_i\gamma'} = -\overline{u_k'u_i}\frac{\partial\overline{\Gamma}'}{\partial x_k'} - \frac{\partial}{\partial x_k'}\overline{u_k'u_i\gamma'}, \tag{26}$$

where x and x' are moving with the velocity $\overline{U}$ and $\overline{U'}$, respectively. We shall investigate the problem in this frame, and present the resulting computations uncorrected for the distortion of correlation functions due to moving the reference locations with a known, radially variable, mean flow. Such corrections, while simple to apply, do not affect the basic structure of the correlations when, as is the case here, the mean flow is constant axially for distances over which the fluctuating fields are correlated.

A natural way to investigate the scalar field just prior to its final period is to approximate the quantities on the right-hand sides of (24)–(26) by the leading terms in their asymptotic time expansions. We already know the asymptotic spatial structure of the turbulent energy tensor $\overline{u_iu_j'}$, the mean scalar field and second moments $\overline{u_i\gamma'}$ and $\overline{\gamma\gamma'}$. In principle, using the techniques developed in § 3 and employing Phillips's solution for the velocity mode, one can also obtain the spatial structure of the triple cross moments $\overline{u_iu_j'\gamma'}$ and $\overline{u_i\gamma\gamma'}$. The computation is tedious and, moreover, introduces new invariants which are integrals along the axis of the initial correlations, e.g. $\overline{m_1(\chi_1)\,m_2(\chi_1')\,C(\chi_1'')}$.

However, one simple situation immediately suggests itself. One notes that, for each of (25) and (26), the triple moment terms on the right-hand side decay more rapidly than those involving lower-order moments. For example, the production $\overline{u_i u_j'}(\partial\overline{\Gamma}'/\partial x_j')$ and the turbulent transport $(\partial/\partial x_j')\overline{u_i u_j' \gamma'}$ decay asymptotically as $(t-t_0)^{-5}$ and $(t-t_0)^{-\frac{11}{2}}$, respectively. Thus there exists a regime prior to the final period in which the dominant phenomenon, apart from diffusive decay, is production of velocity–scalar interaction by the action of the Reynolds stresses on the mean scalar gradient. Moreover, this regime, which we call the penultimate period of decay, is more extensive in the case when the processes of generation of the turbulence and of production of the scalar field are statistically independent. Triple correlation invariants such as $\overline{m_1(\chi_1)\, m_1(\chi_1')\, C(\chi_1'')}$ above, which dominate the asymptotic behaviour of all mixed moments, will then be zero. In this situation every term on the right-hand sides of (24)–(26) will decay more rapidly than is apparent from table 1, with the exception of the production term $-\overline{u_i u_j'}(\partial\overline{\Gamma}'/\partial x_j')$, which is uninfluenced by initial statistical independence.

We may therefore make use of (26) to solve for $\overline{u_i\gamma'}$ in this regime. Since the turbulent energy tensor has been developed in Fourier space, it is useful to rewrite (20) in that form, making use of the assumption of initial statistical independence:

$$\left\{\frac{\partial}{\partial t} + D(k_1^2+k^2) + \nu(k_1^2+k'^2)\right\}\phi_i(k_1,\mathbf{k},\mathbf{k}',t) = -i\int k_j'' J(k'',t)\,\phi_{ij}(k_1,\mathbf{k},\mathbf{k}'-\mathbf{k}'',t)\,d\mathbf{k}'', \tag{27}$$

where $(k_1,\mathbf{k},\mathbf{k}')$ are the Fourier variables analogous to $(r_1,\mathbf{x},\mathbf{x}')$, ϕ_i is the transform of $\overline{u_i\gamma'}$ and J and ϕ_{ij} are given by (14) and (16). The solution of (27) is

$$\begin{aligned}\phi_i(k_1,\mathbf{k},\mathbf{k}',t) = {} & \phi_i(k_1,\mathbf{k},\mathbf{k}',t_1)\exp-\{[D(k_1^2+k^2)+\nu(k_1^2+k'^2)]\,(t-t_1)\}\\ & -i\int_{t_1}^{t}\left\{\int_{k''} k_j'' J(k'',t')\,\phi_{ij}(k_1,\mathbf{k},\mathbf{k}'-\mathbf{k}'',t')\,dk''\right\}\\ & \times\exp-\{[D(k_1^2+k^2)+\nu(k_1^2+k'^2)](t-t')\}\,dt',\end{aligned} \tag{28}$$

where $\phi_i(k_1,\mathbf{k},\mathbf{k}',t_1)$ is initial data.

The first term on the right-hand side of (28) corresponds to the leading term in an asymptotic expansion of ϕ_i. To be consistent with the assumption of no initial statistical correlation of $\mathbf{u}$ and γ, one must have

$$\phi_i(k_1,\mathbf{k},\mathbf{k}',t_1) = 0$$

and

$$\begin{aligned}\phi_i(k_1,\mathbf{k},\mathbf{k}',t) = {} & -i\int_{t_1}^{t}\left\{\int_{k''} k_j'' J(\mathbf{k}'',t')\,\phi_{ij}(k_1,\mathbf{k},\mathbf{k}'-\mathbf{k}'',t')\right\}\\ & \times\exp-\{[D(k_1^2+k^2)+\nu(k_1^2+k'^2)]\,(t-t')\}\,dt'.\end{aligned} \tag{29}$$

Integration of (29) has been achieved using the method of steepest descent for the convolution integral over wavenumber space and by evaluating the time integral analytically. After some effort, which includes replacing the formal singularity in ϕ_{ij} at $t=t_0$ by any finite value, we find, in the limit $t_1\to t_0$,

$$\begin{aligned}\phi_1(K,\mathbf{K},\mathbf{K}',t) \sim {} & \frac{\bar{c}\pi^3\alpha^2[(1/\alpha-2)\,K_1^2+K^2/\alpha+\alpha K'^2]^{\frac{5}{2}}}{\nu[\nu(t-t_0)]^4\,[(K_1^2+K^2)\,(K_1^2+(1-\alpha)^2\,(K'^2)]}\,I_i(K_1,K,K')\\ & \times\exp-\{2K_1^2+K^2+(1-\alpha)\,K'^2\},\end{aligned} \tag{30}$$

where $(K_1, \mathbf{K}, \mathbf{K}')$ are the Fourier variables corresponding to $(\xi_1, \boldsymbol{\xi}, \boldsymbol{\xi}')$,

$$I_2(K_1, \mathbf{K}, \mathbf{K}') = \{G[K_1^6 K_2' - K_1^4 K_3(K_2' K_3 - K_3' K_2)] - F(1-\alpha) K_1^4 K'^2 K_2\}, \quad (31)$$

and

$$I_3(K_1, \mathbf{K}, \mathbf{K}') = \{G[K_1^6 K_3' - K_1^4 K_2(K_3' K_2 - K_2' K_3)] - F(1-\alpha) K_1^4 K'^2 K_3\}. \quad (32)$$

One conclusion which can be drawn immediately from (30) is that the asymptotic decay rate for $\overline{u_i \gamma'}$ is $(t-t_0)^{-4}$, rather than $(t-t_0)^{-3}$ as was obtained in the general case when the **u** field and the concentration field were statistically correlated at their generation.

From (30) for $\overline{u_i \gamma'}$ in the penultimate period of decay, it is possible to solve for the mean concentration. In Fourier space (24) becomes

$$\frac{\partial J}{\partial t} + Dk^2 J = -ik_i \int \phi_i(k_1, \mathbf{k}'', \mathbf{k}-\mathbf{k}'', t)\, d\mathbf{k}'', \quad (33)$$

for which the solution is

$$J(k,t) = J(k,t_1)\exp\{-Dk^2(t-t_1)\}$$
$$-i\int_{t_1}^{t} \left\{\int_{k''} k_i \phi_i(k_1, \mathbf{k}'', \mathbf{k}-\mathbf{k}'', t')\, d\mathbf{k}''\, dk_1\right\} \exp\{-Dk^2(t-t')\}\, dt'. \quad (34)$$

The first term represents the final period of decay result, (14). Hence the modification to the mean field, which we represent by $\delta J(k,t)$, is

$$\delta J(k,t) = -i\int_{t_1}^{t} \left\{\int_{k''} k_i \phi_i(k_1, \mathbf{k}'', \mathbf{k}-\mathbf{k}'', t')\, d\mathbf{k}''\, dk_1\right\} \exp\{-Dk^2(t-t')\}\, dt', \quad (35)$$

where ϕ_i is given by (30).

The convolution integral in (35) has been evaluated approximately using the method of steepest descent, and the time integration has been carried out analytically. This evaluation of $\delta J(k,t)$ is outlined in the appendix, as a typical example of the analysis we have used in this section. We find, for $Pr \leqslant 1$ and K large (see (A 3) and (A 5)),

$$\delta J(k,t) \sim -\frac{|K|^9 F \exp\{-[(1-\alpha)/(2-\alpha)] K^2\}}{[\nu(t-t_0)]^5}. \quad (36)$$

The very rapid decay rate $(t-t_0)^{-5}$ indicates that the mean field is highly insensitive to velocity–scalar interactions in the asymptotic time regime, since the scalar mean itself decays asymptotically as $(t-t_0)^{-1}$. Analogous computations can be done for the scalar intensity spectrum $\phi(k_1, \mathbf{k}, \mathbf{k}', t)$ in the penultimate period of decay of a wake, in which the scalar and velocity fluctuations are generated by statistically independent processes. The appropriate equation is, from (5),

$$\{(\partial/\partial t) + D(k^2 + k'^2)\}\phi(k_1, \mathbf{k}, \mathbf{k}', t)$$
$$= -i\int k_j'' J(k'', t)\,[\phi_j(k_1, \mathbf{k}, \mathbf{k}'-\mathbf{k}'', t') + \phi_j(k_1, \mathbf{k}', \mathbf{k}-\mathbf{k}'', t')]\, d\mathbf{k}''. \quad (37)$$

Defining $\delta\phi$ to be the difference between the scalar field obtained by solving (37) and the final period decay result (15), we can show that

$$\delta\phi \sim \frac{[\nu(t-t_0)]^{-\frac{9}{2}}}{(1-\alpha)^2(1+\alpha)}$$

$$\left\{ K_j' \frac{\left[\left(\frac{1}{\alpha}-2\right)K_1^2+\frac{1}{2}K^2+\frac{\alpha}{(1+\alpha)^2}K'^2\right]^{\frac{5}{2}} I_j\left(K_1, \mathbf{K}, \frac{1}{1+\alpha}\mathbf{K}'\right) \exp\left\{-\left(2K_1^2+K^2+\frac{1-\alpha}{\alpha(1+\alpha)}K'^2\right)\right\}}{[K_1^2+K'^2]\left[K_1^2+\left(\frac{1-\alpha}{2-\alpha}\right)^2 K'^2\right]\left[\frac{2(1-2\alpha)}{\alpha}K^2+\frac{1-\alpha}{\alpha(1+\alpha)}K'^2\right]} \right.$$

$$\left. + K_j \frac{\left[\left(\frac{1}{\alpha}-2\right)K_1^2+\frac{1}{2}K'^2+\frac{\alpha}{(1+\alpha)^2}K^2\right]^{\frac{5}{2}} I_j\left(K_1, \mathbf{K}', \frac{1}{1+\alpha}\mathbf{K}\right) \exp\left\{-\left(2K_1^2+K'^2+\frac{1-\alpha}{\alpha(1+\alpha)}K^2\right)\right\}}{[K_1^2+K^2]\left[K_1^2+\left(\frac{1-\alpha}{2-\alpha}\right)^2 K^2\right]\left[\frac{2(1-2\alpha)}{\alpha}K'^2+\frac{1-\alpha}{\alpha(1+\alpha)}K^2\right]} \right\}, \tag{38}$$

where there is a restriction that $0{\cdot}3 < \alpha$ or $0{\cdot}43 < Pr \leqslant 1$.

The temporal behaviour of the perturbation to scalar intensity in the penultimate period is as $(t-t_0)^{-\frac{9}{2}}$, which is significantly more rapid a decay than the final period scalar intensity result $(t-t_0)^{-\frac{5}{2}}$ displayed in table 1. If the assumption of initial statistical independence had not been made, the structures for $\delta\phi$ and δJ in the penultimate period would be different from those given by (36) and (38), since the proper ϕ_i to have used would have been the Fourier transforms of (21)–(23), rather than (29). The numerical computation is formidable, and we have not attempted it. However, it is easy to see that the temporal behaviour of the perturbations would become

$$\delta J \sim (t-t_0)^{-4}, \quad \delta\phi \sim (t-t_0)^{-\frac{7}{2}}.$$

5. Numerical results

Numerical computations have been made from the various results reported in §§3 and 4. In all cases we have expressed the data in normalized real space rather than in Fourier space. The ordinate scale is arbitrary since absolute magnitudes are strongly dependent on time, as is evident from table 1. Furthermore, there are undetermined constants such as F, G and E, whose values depend on the details of the generating processes. For the purposes of computation these invariants are taken to have equal value. All functions are properly axisymmetric and, unless otherwise stated, the Prandtl number is unity.

Figure 1 is a plot of the final period behaviour of $\overline{u_2\gamma}$ and $\overline{u_2\gamma^2}$, and it was discussed at the end of §3. Only correlations at coincident points have been measured in an axisymmetric heated wake (Freymuth & Uberoi 1973), and these have been at Reynolds numbers much larger than those that should characterize the final period of decay. Nevertheless, the results of figure 1 show a qualitative similarity to the reported measurements. Perhaps this result is explained by noting that $\overline{u\gamma}$ is linked to the mean concentration field through (4), and the mean itself exhibits extraordinarily strong insensitivity to perturbations

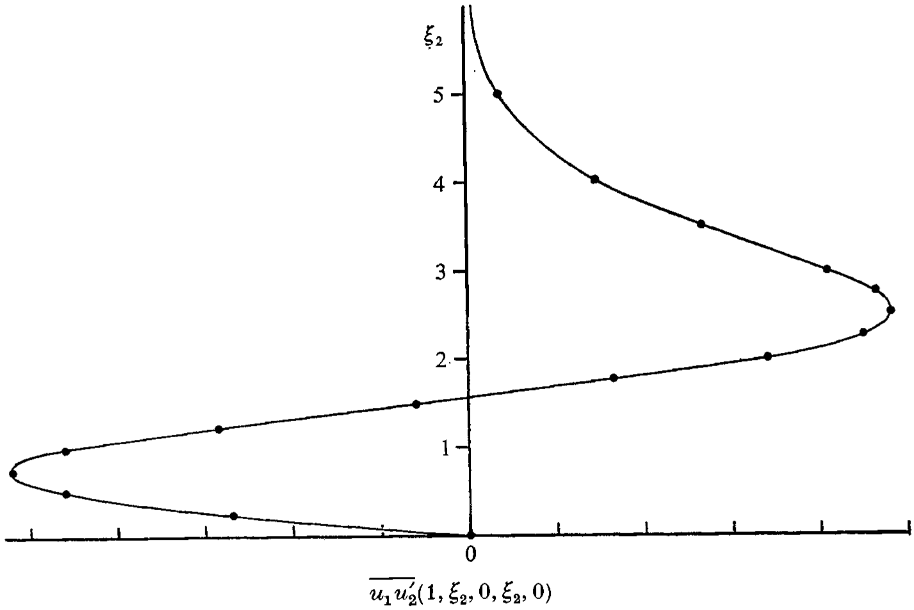

FIGURE 2. Energy tensor component $\overline{u_1u_2'}(1, \xi_2, 0, \xi_2, 0)$ in the final period.

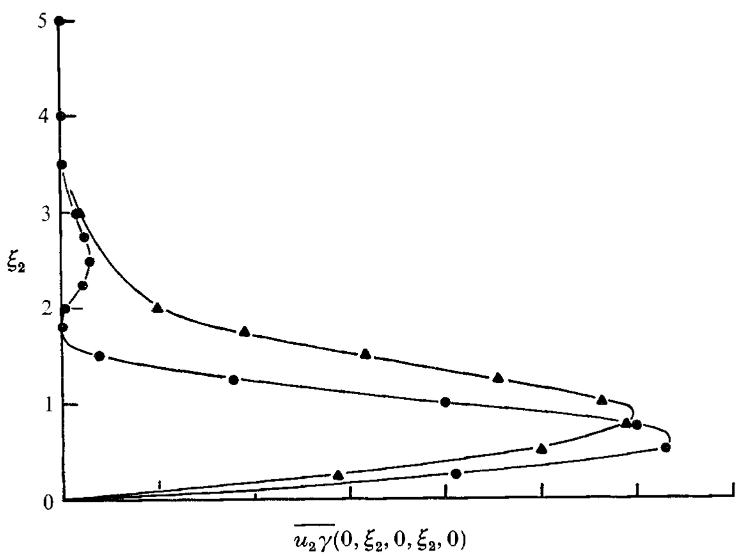

FIGURE 3. Velocity–scalar correlation $\overline{u_2\gamma}(0, \xi_2, 0, \xi_2, 0)$ in the penultimate period: ●, $Pr = 1$; ▲, $Pr = 0{\cdot}01$.

in the asymptotic regimes both experimentally and theoretically from §4. In figure 2 we reproduce a computation of one component of the energy tensor $\overline{u_1u_2'}(1, \xi_2, 0, \xi_2, 0)$ obtained by a Fourier transform of ϕ_{12} in (16), which is of course a final period result.

Figures 3 and 4 display features of the velocity–scalar correlation which result, in the penultimate period of decay, from interactions between mean scalar

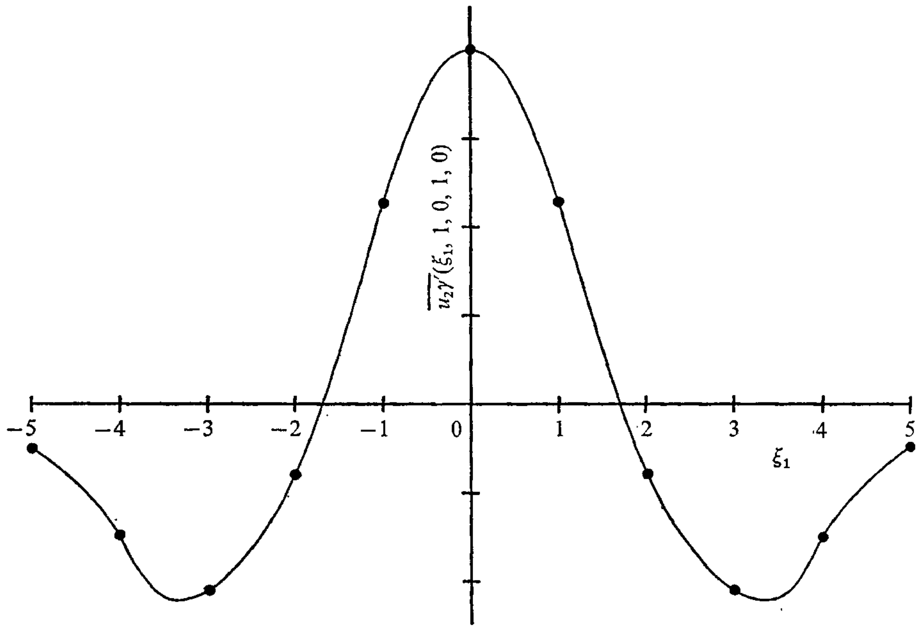

FIGURE 4. Velocity–scalar correlation $\overline{u_2\gamma'}(\xi_1, 1, 0, 1, 0)$ in the penultimate period.

gradients and non-coincident Reynolds stress tensors. In figure 3 $\overline{u_2\gamma}(0, \xi_2, 0, \xi_2, 0)$ is plotted with Prandtl number as parameter and it exhibits the expected behaviour. We note by axisymmetry that $\overline{u_2\gamma}$ is an antisymmetric function of ξ_2 and, by axial homogeneity, $\overline{u_2\gamma'}(\xi_1, \xi_2, 0, \xi_2, 0)$ is a symmetric function of ξ_1. It is to be noted that $\overline{u_2\gamma}$ in the penultimate period is also generally similar to that measured in the more energetic regions of the wake. The initial assumption that the turbulence and scalar fields are in a uniform state of decay is supported by figure 4, in which the longitudinal correlation $\overline{u_2\gamma'}(\xi_1, 1, 0, 1, 0)$ is plotted. We had difficulty computing for values of $\xi_1 > 5$, but it seems that the longitudinal correlation distance is of the same order as the width of the wake in figure 2.

In figures 5 and 6 we show aspects of the structure of the perturbation to the concentration correlation in the penultimate period, produced by interactions between mean scalar gradients and velocity–scalar correlations In figure 5 the dependence of $\delta\overline{\gamma\gamma'}$ on lateral separation is depicted by plotting

$$\delta\overline{\gamma\gamma'}(0, \xi_2, 0, \xi_2 + \Delta\xi_2, 0)$$

against ξ_2 with $\Delta\xi_2$ as parameter. The curve $\Delta\xi_2 = 0$ represents the perturbation to scalar intensity, which is symmetric in ξ_2 and which has the same general form as is needed to reproduce measured profiles of $\overline{\gamma^2}$ (Freymuth & Uberoi 1973, figure 3). This is not surprising. It seems likely from the already established results that $\overline{u_2\gamma}$ and $\overline{\Gamma}$ are qualitatively similar to their measured values at moderate Reynolds number. Figure 6 represents the longitudinal correlation $\delta\overline{\gamma\gamma'}(\xi_1, 1, 0, 1, 0)$ for various positive ξ_1. By definition $\delta\overline{\gamma\gamma'}$ is a symmetric function of ξ_1. We have not plotted $\delta\overline{\Gamma}(\xi, t)$, which is obtained as (A 6), since the extra-

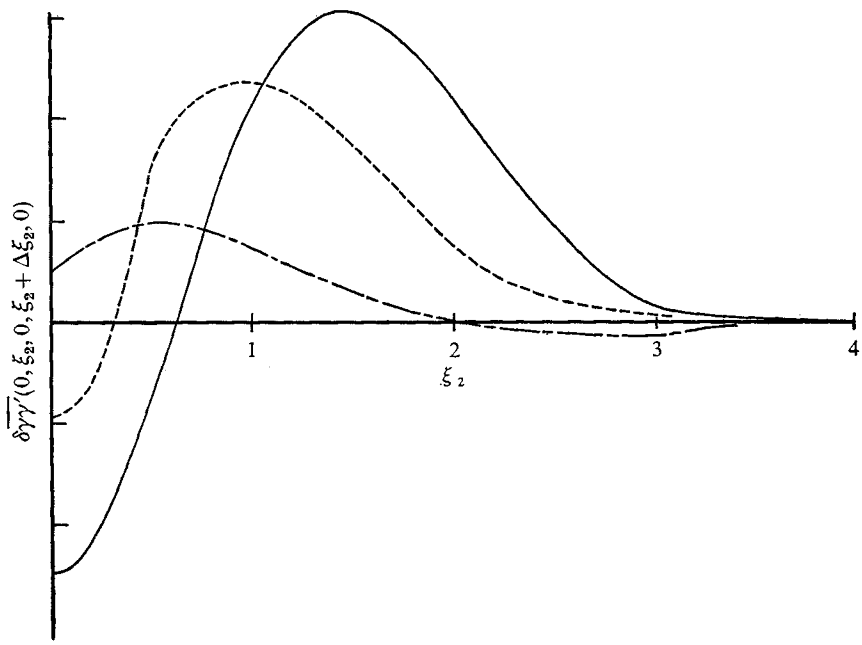

FIGURE 5. Scalar correlation perturbation $\delta\overline{\gamma\gamma'}(0, \xi_2, 0, \xi_2+\Delta\xi_2, 0)$ in the penultimate period: ——, $\Delta\xi_2 = 0$; ----, $\Delta\xi_2 = 1$; — — — —, $\Delta\xi_2 = 2$.

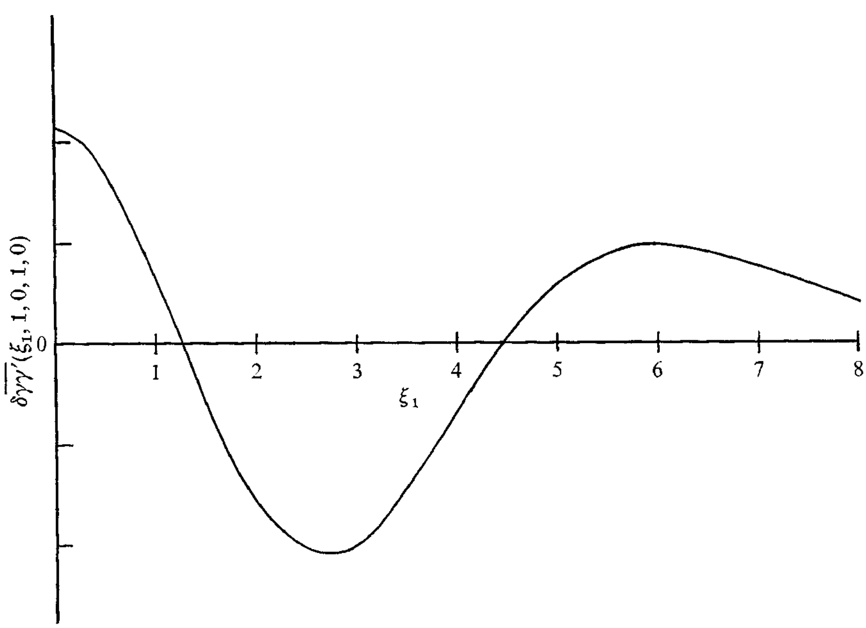

FIGURE 6. Scalar correlation perturbation $\delta\overline{\gamma\gamma'}(\xi_1, 1, 0, 1, 0)$ in the penultimate period.

ordinarily rapid time decay of this function as compared to $\overline{\Gamma}(\xi, t)$ makes it unlikely that it would be observed experimentally.

The wake structure, and hence the scalar field structure, in the penultimate period will be altered if the net momentum of the body is zero. The velocity field in this case has been obtained (Phillips 1955), and the same methods as used in this paper could be employed to deduce the corresponding asymptotic scalar structure.

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Appendix. Evaluation of mean scalar field modification δJ

$$\frac{\partial J}{\partial t} + Dk^2 J = -i \int_{-\infty}^{+\infty} \int_{k''} k_j \phi_j(k_1, \mathbf{k}'', \mathbf{k} - \mathbf{k}'')\, d\mathbf{k}''\, dk_1, \tag{A 1}$$

where ϕ_j is given by (30). The convolution integral over the two-dimensional vector $\mathbf{k}''$ is carried out using the approximate method of steepest descent (Heading 1962). A typical integral is of the form

$$I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(k_1, \mathbf{k}, \mathbf{k}_2'', \mathbf{k}_3'') \exp\{f(k_1, \mathbf{k}, k_2'', k_3'')\}\, dk_2''\, dk_3'',$$

which can be approximated by

$$I = -\frac{2\pi}{\lambda} g(k_1, \mathbf{k}, \mathbf{k}^{*\prime\prime}) \exp f(k_1, \mathbf{k}, \mathbf{k}^{*\prime\prime}),$$

where $\mathbf{k}^{*\prime\prime}$ is that vector value of $\mathbf{k}''$ which maximizes f;

$$\lambda = \frac{\partial^2 f}{\partial k_2''^2} = \frac{\partial^2 f}{\partial k_3''^2}.$$

Consequently, we find that the right-hand side of (A 1), say Σ, can be written approximately as

$$\Sigma(K, t) \sim \frac{1}{[\nu(t-t_0)]^6} \Lambda(K) \exp\left\{-\left(\frac{1-\alpha}{2-\alpha}\right) K^2\right\}, \tag{A 2}$$

where

$$\Lambda(K) = \int_{-\infty}^{+\infty} \frac{\left[\left(\frac{1}{\alpha} - 2\right) K_1^2 + \left(\frac{1 - 2\alpha + 2\alpha^2}{\alpha(2-\alpha)}\right) K^2\right]^{\frac{5}{2}} \left\{G K_1^6 K^2 - F \left(\frac{1-\alpha}{2-\alpha}\right)^2 K_1^4 K^4\right\} \exp\{-2K_1^2\, dK_1\}}{\left[K_1^2 + \left(\frac{1-\alpha}{2-\alpha}\right) K^2\right] \left[K_1^2 + \left(\frac{1-\alpha}{2-\alpha}\right)^2 K^2\right]} \tag{A 3}$$

and $\alpha \leqslant 0{\cdot}5$. Define

$$\tau' = (t' - t_0)\nu \quad \text{and} \quad \tau = (t - t_0)\nu$$

then $$\delta J(k, t) \sim \exp\left\{-\left(\frac{1-\alpha}{\alpha}\right) k^2 \tau\right\} \int_0^{\tau} \frac{\Lambda(k^2 \tau')}{\tau'^6} \exp\left(\frac{2(1-\alpha)^2}{\alpha(2-\alpha)} k^2 \tau'\right) d\tau',$$

$$\text{or}\quad \delta J(k,t) \sim \frac{K^{10}}{[\nu(t-t_0)]^5} \exp\left\{-\left(\frac{1-\alpha}{\alpha}\right)K^2\right\} \int_0^K \frac{\Lambda(K')}{K'^6} \exp\left\{\frac{2(1-\alpha)^2}{\alpha(2-\alpha)} K'^2\right\} dK'. \tag{A 4}$$

Using the asymptotic result (Abramowitz & Stegun 1965),

$$\lim_{K\to\infty} \int_C^K \frac{e^y}{y}\, dy \sim \frac{e^K}{K},$$

where C is any positive constant, and replacing the singularity at $K' = 0$ by any integrable behaviour there, we obtain for large K

$$\delta J(k,t) \sim -\frac{k^{10} \exp\{-[(1-\alpha)/(2-\alpha)]\, k^2\nu(t-t_0)\}}{[k^2\nu(t-t_0)]^{\frac{1}{2}}},$$

or

$$\delta J(k,t) \sim -\frac{|K|^9 F \exp\{-[(1-\alpha)/(2-\alpha)]\, K^2\}}{[\nu(t-t_0)]^5}. \tag{A 5}$$

The inverse Fourier transform of (A 5) is readily obtained (Gröbner & Hofreiter 1961):

$$\delta\overline{\Gamma} = -\frac{A(\alpha)}{[\nu(t-t_0)]^5} M\left(5\tfrac{1}{2}, 1, -\frac{(2-\alpha)}{4(1-\alpha)}\xi^2\right), \tag{A 6}$$

where $M(a,b,z)$ is a confluent hypergeometric function known as Kummer's function (Abramowitz & Stegun 1965), and $A(\alpha)$ is a function of the Prandtl number only. For small values of ξ^2, a limit consistent with the process by which (A 5) approximated (A 4), we find

$$\delta\overline{\Gamma} = -\frac{A(\alpha)}{[\nu(t-t_0)]^5}\left[1 - \frac{11}{8}\frac{2-\alpha}{1-\alpha}\xi^2 + O(\xi^4)\right]. \tag{A 7}$$

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